Lecture 4 - Sep. 17

Math Review

Logical ($\wedge$) vs. Programming (&&)
False Range $\neg R(x)$ vs. Empty Array
Proof Strategies of Quantifiers

Announcements/Reminders

Lab2

- Lab1 due this Friday at noon.
- Scheduled lab sessions tomorrow.
- Study along with the Math Review lecture notes.

Logical Operator vs. Programming Operator

p	q	$p \wedge q$	$p \vee q$
true	true	true	true
true	false	false	true
false	true	false	true
false	false	false	false

short circuit

p && q

↳ evaluate from L to R

↳ if L evaluates to false

↳ evaluation of R is skipped

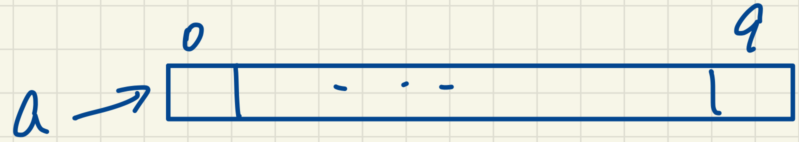
Q. Are the $\wedge$ and $\vee$ operators equivalent to, respectively, && and || in Java?

$p \wedge q$

① both p and q are well-defined

② "evaluate" both sides separately.

Accessing Array



$a.length == 10$

① ② ③

(1) $i < a.length \;\&\&\; \boxed{a[i] > 10} \;\&\&\; i >= 0$

Math

$i < a.length \land$
$i >= 0 \land$
$a[i] > 10$

$i < a.length \land$
$i >= 0$
$\Rightarrow$
$a[i] > 10$

Ill-defined expression

say $i == -2$ (fail)

① $i < a.length$ (T)

-2

② $a[-2] > 10$

crash!

say $i == 100$

① $100 < \underset{10}{a.length}$ (F)

$a[i] > 10$ is skipped and expression just evaluates to false.

✓ (2) $i < a.length \;\&\&\; i >= 0 \;\&\&\; a[i] > 10$

guarding constraint.

Predicate Logic: Quantifiers

$\forall i \cdot Q(i)$
$\exists i \cdot Q(i)$

- syntax
- base cases in programming

$\forall i \bullet \boxed{R(i)} \Rightarrow P(i)$

false

what if empty range?

true (zero of $\Rightarrow$)

$\neg R(i)$

$\exists i \bullet \boxed{R(i)} \wedge P(i)$

false

false.

what if

$R(i) \equiv$ false ?

can you find a # in a that shows otherwise?

input

```
boolean allPositive(int[] a){
  if(a.length == 0){ return true }
}
```

① ∵ no witness can be found in empty array to prove otherwise

```
boolean somePositive(int[] a){
  if (a.length == 0) { return false }
}
```

② ∵ no witness can be found in empty array to prove.

$\mathbb{Z}$ set of integers

$-\infty, \cdots, -1, 0, 1, \cdots, +\infty$

$\mathbb{N}$ set of natural numbers

$0, 1, 2, \cdots, +\infty$

$\mathbb{N}_1$ $1, 2, \cdots, +\infty$

$$\forall \bar{i}, \bar{j} \cdot \bar{i} \in \mathbb{N} \wedge \bar{j} \in \mathbb{Z} \Rightarrow P(\bar{i}, \bar{j})$$

need to consider <u>all</u> combinations of $(\bar{i}, \bar{j})$.

Logical Quantifiers: Examples

$\forall i \bullet i \in \mathbb{N} \Rightarrow \boxed{i \geq 0}$ T

$i \in 0, 1, \ldots, +\infty$

$\forall i \bullet i \in \mathbb{Z} \Rightarrow i \geq 0$ false witness : -1

$-1 \in \mathbb{Z} \Rightarrow -1 \geq 0$

T F

F

$\forall i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \Rightarrow i < j \vee i > j$

False witness $i = j$

$i = 1$
$j = 1$

$\exists i \bullet i \in \mathbb{N} \wedge i \geq 0$ T witness: 0

$\exists i \bullet i \in \mathbb{Z} \wedge i \geq 0$ T witness : 3

$\exists i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \wedge (i < j \vee i > j)$ T witness : $i = 2$
$j = 3$

Logical Quantifiers: Examples

Goal: show $R(\bar{i}) \Rightarrow P(\bar{i}) \equiv true$

How to $\boxed{\text{prove}}$ $\forall\, i \bullet R(i) \Rightarrow P(i)$?

trivial (1) show $\neg R(\bar{i})$ ∵ zero of $\Rightarrow$: $false \Rightarrow P \equiv true$

harder (2) show $R(\bar{i})$, $P(\bar{i})$ (e.g. all elements in a non-empty array are positive)

How to $\boxed{\text{prove}}$ $\exists\, i \bullet R(i) \wedge P(i)$?

Goal: show $R(\bar{i}) \wedge P(\bar{i}) \equiv true$.

not hard (1) give a witness $\bar{j}$, s.t. $R(\bar{j})$ and $P(\bar{j})$

Goal: show $R(\bar{i}) \Rightarrow P(\bar{i}) \equiv false$

How to $\boxed{\text{disprove}}$ $\forall\, i \bullet R(i) \Rightarrow P(i)$?

not hard. (1) show $R(\bar{i})$, $\neg P(\bar{i})$

give a witness $\bar{j}$, s.t. $R(\bar{j})$ but $\neg P(\bar{j})$

How to $\boxed{\text{disprove}}$ $\exists\, i \bullet R(i) \wedge P(i)$?

Goal: show $R(\bar{i}) \wedge P(\bar{i})$ is false.

trivial (1) show $\neg R(\bar{i})$: $false \wedge P \equiv false$

harder. (2) for those $\bar{i}$ satisfying $R(\bar{i})$, it's the case $\neg P(\bar{i})$.

Prove/Disprove Logical Quantifications

all x in $1..10$ satisfies this

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 0.$

$x \in 1..10$ (non-empty range $R(x)$ not false).

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 1.$

Exercise.

- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 1.$

By choosing a witness 1, which:

$T \; (1 \in \mathbb{Z} \land 1 \leq 1 \leq 10) \land \overset{F}{(1 > 1)} \;\; F$

non-empty range. witness: 2

- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 10?$

Exercise.

not sufficient by just one witness to disprove $\exists$.

Logical Quantifications: Conversions

$R(x)$: $x \in$ 3342_class

$P(x)$: x receives A+

$\forall x \cdot Q(x) \iff \neg \exists x \cdot \neg Q(x)$

$(\forall X \bullet R(X) \Rightarrow P(X)) \iff \neg(\exists X \bullet R \wedge \neg P)$

$\forall x \cdot R(x) \Rightarrow P(x)$

$\iff \{ \forall x \cdot Q(x) \iff \neg \exists x \cdot \neg \boxed{Q(x)} \}$

$\neg \exists x \cdot \neg (R(x) \Rightarrow P(x))$

$\iff \{ p \Rightarrow q \equiv \neg p \vee q \}$

$\neg \exists x \cdot \neg(\neg R(x) \vee P(x))$

$\iff \{ \neg(p \vee q) \equiv \neg p \wedge \neg q, \neg(\neg p) \equiv p \}$

$(\exists X \bullet R \wedge P) \iff \neg(\forall X \bullet R \Rightarrow \neg P)$

$\neg \exists x \cdot R(x) \wedge \neg P(x)$

Exercise

De Morgan